Noether conservation laws in higher-dimensional Chern–Simons theory

GIOVANNI GIACHETTA,[1] LUIGI MANGIAROTTI[2]

Department of Mathematics and Informatics, University of Camerino, 62032 Camerino (MC) Italy

GENNADI SARDANASHVILY[3]

Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Though a global Chern–Simons $(2k-1)$-form is not gauge-invariant, this form seen as a Lagrangian of higher-dimensional gauge theory leads to the conservation law of a modified Noether current.

One usually considers Chern–Simons (henceforth CS) gauge theory on a principal bundle over a three-dimensional manifold whose Lagrangian is the local CS form derived from the local transgression formula for the second Chern characteristic form. This Lagrangian fails to be globally defined, unless a principal bundle is trivial (e.g., if its structure group is simply connected [4]). Though the local CS Lagrangian is not gauge-invariant, it leads to the (local) conservation law of the modified Noether current [2, 5, 7]. This result is extended to the global three-dimensional CS theory [1, 3]. Its Lagrangian is well defined, but depends on a background gauge potential. Therefore, it is gauge-covariant, but not gauge-invariant. At the same time, the corresponding Euler–Lagrange operator is gauge-invariant, and the above mentioned gauge conservation law takes place. We aim to show that any higher-dimensional CS theory admits such a conservation law.

There are different approaches to the study of Lagrangian conservation laws. We use the so called first variational formula, which enables one to obtain conservation laws if a symmetry is broken [5, 6, 7].

Let us consider a first order field theory on a fibre bundle $Y \to X$ over an n-dimensional smooth manifold X. Its configuration space is the first order jet manifold J^1Y of sections of $Y \to X$. Given bundle coordinates (x^λ, y^i) on a fibre bundle $Y \to X$, its first and second

[1]E-mail: giovanni.giachetta@unicam.it

[2]E-mail: luigi.mangiarotti@unicam.it

[3]E-mail: sard@grav.phys.msu.su

order jet manifolds J^1Y and J^2Y are endowed with the adapted coordinates $(x^\lambda, y^i, y^i_\mu)$ and $(x^\lambda, y^i, y^i_\mu, y^i_{\lambda\mu})$, respectively. One can think of y^i_μ and $y^i_{\lambda\mu}$ as being coordinates of first and second derivatives of dynamic variables. We use the notation $\omega = d^n x$ and $\omega_\lambda = \partial_\lambda \rfloor \omega$.

A first order Lagrangian of field theory on $Y \to X$ is defined as a density

$$L = \mathcal{L}(x^\mu, y^j, y^j_\mu)\omega \tag{1}$$

on the first order jet manifold J^1Y of $Y \to X$. Given a Lagrangian L (1), the corresponding Euler–Lagrange operator reads

$$\delta L = \delta_i \mathcal{L} \theta^i \wedge \omega = (\partial_i \mathcal{L} - d_\lambda \partial^\lambda_i)\mathcal{L}\theta^i \wedge \omega, \tag{2}$$

where $\theta^i = dy^i - y^i_\lambda dx^\lambda$ are contact forms and

$$d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_{\lambda\mu}\partial^\mu_i$$

are the total derivatives, which yield the total differential $d_H\varphi = dx^\lambda \wedge d_\lambda \varphi$ acting on exterior forms on J^1Y. The kernel $\mathrm{Ker}\,\delta L \subset J^2Y$ of the Euler–Lagrange operator (2) defines the Euler–Lagrange equations

$$\delta_i \mathcal{L} = (\partial_i \mathcal{L} - d_\lambda \partial^\lambda_i)\mathcal{L} = 0. \tag{3}$$

A Lagrangian L (1) is said to be variationally trivial if $\delta L = 0$. This property holds iff $L = h_0(\varphi)$, where φ is a closed n-form on Y and h_0 is the horizontal projection

$$h_0(dx^\lambda) = dx^\lambda, \qquad h_0(dy^i) = y^i_\lambda dx^\lambda, \qquad h_0(dy^i_\mu) = y^i_{\lambda\mu} dx^\lambda.$$

The relation $d_H \circ h_0 = h_0 \circ d$ holds.

To obtain Noether conservation laws, one considers local one-parameter groups of vertical bundle automorphisms (gauge transformations) of $Y \to X$. Their infinitesimal generators are vertical vector fields $u = u^i(x^\mu, y^j)\partial_i$ on $Y \to X$ whose prolongation onto J^1Y reads

$$J^1u = u^i\partial_i + d_\lambda u^i \partial^\lambda_i. \tag{4}$$

A Lagrangian L is invariant under a one-parameter group of gauge transformations generated by a vector field u iff its Lie derivative

$$\mathbf{L}_{J^1u}L = J^1u \rfloor dL = (u^i\partial_i\mathcal{L} + d_\lambda u^i \partial^\lambda_i \mathcal{L})\omega \tag{5}$$

along J^1u vanishes. The first variational formula provides the canonical decomposition

$$\mathbf{L}_{J^1u}L = u\rfloor\delta L + d_H(u\rfloor H_L) = u^i\delta_i\mathcal{L}\omega + d_\lambda(u^i\partial_i^\lambda\mathcal{L})\omega, \tag{6}$$

where $H_L = \mathcal{L}\omega + \partial_i^\lambda\mathcal{L}\theta^i\wedge\omega_\lambda$ is the Poincaré–Cartan form of L, and

$$\mathfrak{J}_u = u\rfloor H_L = \mathfrak{J}_u^\lambda\omega_\lambda = u^i\partial_i^\lambda\mathcal{L}\omega_\lambda \tag{7}$$

is the symmetry current along u. On the shell (3), the first variational formula (6) leads to the weak equality

$$\mathbf{L}_{J^1u}L \approx -d_H\mathfrak{J}_u, \qquad u^i\partial_i\mathcal{L} + d_\lambda u^i\partial_i^\lambda\mathcal{L} \approx d_\lambda(u^i\partial_i^\lambda\mathcal{L}). \tag{8}$$

If $\mathbf{L}_{J^1u}L = 0$, we obtain the Noether conservation law

$$0 \approx d_H\mathfrak{J}_u \tag{9}$$

of the symmetry current $\mathfrak{J}_u$ (7). If the Lie derivative (5) reduces to the total differential

$$\mathbf{L}_{J^ru}L = d_H\sigma, \tag{10}$$

then the weak equality (8) takes the form

$$0 \approx d_H(\mathfrak{J}_u - \sigma), \tag{11}$$

regarded as a conservation law of the modified symmetry current $\overline{\mathfrak{J}} = \mathfrak{J}_u - \sigma$.

Now, let us turn to gauge theory of principal connections on a principal bundle $P\to X$ with a structure Lie group G. Let J^1P be the first order jet manifold of $P\to X$ and

$$C = J^1P/G \to X \tag{12}$$

the quotient of P with respect to the canonical action of G on P [6, 7]. There is one-to-one correspondence between the principal connections on $P\to X$ and the sections of the fibre bundle C (12), called the connection bundle. Given an atlas Ψ of P, the connection bundle C is provided with bundle coordinates (x^λ, a^r_μ) such that, for any its section A, the local functions $A^r_\mu = a^r_\mu\circ A$ are coefficients of the familiar local connection form. From the physical viewpoint, A is a gauge potential.

The infinitesimal generators of one-parameter groups of gauge transformations of the principal bundle P are G-invariant vertical vector fields on P. There is one-to-one correspondence between these vector fields and the sections of the quotient $V_GP = VP/G\to X$

of the vertical tangent bundle VP of $P \to X$ with respect to the canonical action of G on P. The typical fibre of V_GP is the right Lie algebra $\mathfrak{g}$ of the Lie group G, acting on this typical fibre by the adjoint representation. Given an atlas Ψ of P and a basis $\{\epsilon_r\}$ for the Lie algebra $\mathfrak{g}$, we obtain the fibre bases $\{e_r\}$ for V_GP. If $\xi = \xi^p e_p$ and $\eta = \eta^q e_q$ are sections of $V_GP \to X$, their bracket is

$$[\xi, \eta] = c^r_{pq}\xi^p\eta^q e_r,$$

where c^r_{pq} are the structure constants of $\mathfrak{g}$. Note that the connection bundle C (12) is an affine bundle modelled over the vector bundle $T^*X \otimes V_GP$, and elements of C are represented by local V_GP-valued 1-forms $a^r_\mu dx^\mu \otimes e_r$. The infinitesimal generators of gauge transformations of the connection bundle $C \to X$ are vertical vector fields

$$\xi_C = (\partial_\mu\xi^r + c^r_{pq}a^p_\mu\xi^q)\partial^\mu_r. \tag{13}$$

The connection bundle $C \to X$ admits the canonical V_GP-valued 2-form

$$\mathfrak{F} = (da^r_\mu \wedge dx^\mu + \frac{1}{2}c^r_{pq}a^p_\lambda a^q_\mu dx^\lambda \wedge dx^\mu) \otimes e_r, \tag{14}$$

which is the curvature of the canonical connection on the principal bundle $C \times P \to C$ [6]. Given a section A of $C \to X$, the pull-back

$$F_A = A^*\mathfrak{F} = \frac{1}{2}F^r_{\lambda\mu}dx^\lambda \wedge dx^\mu \otimes e_r, \qquad F^r_{\lambda\mu} = \partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq}A^p_\lambda A^q_\mu, \tag{15}$$

of $\mathfrak{F}$ onto X is the strength form of a gauge potential A.

Turn now to the CS forms. Let $I_k(\epsilon) = b_{r_1\ldots r_k}\epsilon^{r_1}\cdots\epsilon^{r_k}$ be a G-invariant polynomial of degree $k > 1$ on the Lie algebra $\mathfrak{g}$ written with respect to its basis $\{\epsilon_r\}$, i.e.,

$$\sum_j b_{r_1\ldots r_k}\epsilon^{r_1}\cdots c^{r_j}_{pq}\epsilon^p\cdots\epsilon^{r_k} = kb_{r_1\ldots r_k}c^{r_1}_{pq}\epsilon^p\epsilon^{r_2}\cdots\epsilon^{r_k} = 0.$$

Let us associate to $I(\epsilon)$ the closed gauge-invariant $2k$-form

$$P_{2k}(\mathfrak{F}) = b_{r_1\ldots r_k}\mathfrak{F}^{r_1} \wedge \cdots \wedge \mathfrak{F}^{r_k} \tag{16}$$

on C. Let A be a section of $C \to X$. Then, the pull-back

$$P_{2k}(F_A) = A^*P_{2k}(\mathfrak{F}) \tag{17}$$

of $P_{2k}(\mathfrak{F})$ is a closed characteristic form on X. Recall that the de Rham cohomology of C equals that of X since $C \to X$ is an affine bundle. It follows that $P_{2k}(\mathfrak{F})$ and $P_{2k}(F_A)$ possess the same cohomology class

$$[P_{2k}(\mathfrak{F})] = [P_{2k}(F_A)] \tag{18}$$

for any principal connection A. Thus, $I_k(\epsilon) \mapsto [P_{2k}(F_A)] \in H^*(X)$ is the familiar Weil homomorphism.

Let B be a fixed section of the connection bundle $C \to X$. Given the characteristic form $P_{2k}(F_B)$ (17) on X, let the same symbol stand for its pull-back onto C. By virtue of the equality (18), the difference $P_{2k}(\mathfrak{F}) - P_{2k}(F_B)$ is an exact form on C. Moreover, similarly to the well-known transgression formula on a principal bundle P, one can obtain the following transgression formula on C:

$$P_{2k}(\mathfrak{F}) - P_{2k}(F_B) = d\mathfrak{S}_{2k-1}(B), \tag{19}$$

$$\mathfrak{S}_{2k-1}(B) = k\int_0^1 \mathfrak{P}_{2k}(t,B)dt, \tag{20}$$

$$\mathfrak{P}_{2k}(t,B) = b_{r_1\ldots r_k}(a^{r_1}_{\mu_1} - B^{r_1}_{\mu_1})dx^{\mu_1}\wedge \mathfrak{F}^{r_2}(t,B)\wedge\cdots\wedge\mathfrak{F}^{r_k}(t,B),$$

$$\mathfrak{F}^{r_j}(t,B) = [d(ta^{r_j}_{\mu_j} + (1-t)B^{r_j}_{\mu_j})\wedge dx^{\mu_j} +$$
$$\frac{1}{2}c^{r_j}_{pq}(ta^p_{\lambda_j} + (1-t)B^p_{\lambda_j})(ta^q_{\mu_j} + (1-t)B^q_{\mu_j})dx^{\lambda_j}\wedge dx^{\mu_j}]\otimes e_r.$$

Its pull-back by means of a section A of $C \to X$ gives the transgression formula

$$P_{2k}(F_A) - P_{2k}(F_B) = dS_{2k-1}(A,B)$$

on X. For instance, if $P_{2k}(F_A)$ is the characteristic Chern $2k$-form, then $S_{2k-1}(A,B)$ is the familiar CS $(2k-1)$-form. Therefore, we agree to call $\mathfrak{S}_{2k-1}(B)$ (20) the CS form on the connection bundle C. In particular, one can choose the local section $B=0$. Then, $\mathfrak{S}_{2k-1} = \mathfrak{S}_{2k-1}(0)$ is the local CS form. Let $S_{2k-1}(A)$ denote its pull-back onto X by means of a section A of $C \to X$. Then, the CS form $\mathfrak{S}_{2k-1}(B)$ admits the decomposition

$$\mathfrak{S}_{2k-1}(B) = \mathfrak{S}_{2k-1} - S_{2k-1}(B) + dK_{2k-1}(B). \tag{21}$$

Let J^1C be the first order jet manifold of the connection bundle $C \to X$ equipped with the adapted coordinates $(x^\lambda, a^r_\mu, a^r_{\lambda\mu})$. Let us consider the pull-back of the CS form (20) onto J^1C denoted by the same symbol $\mathfrak{S}_{2k-1}(B)$, and let

$$\mathcal{S}_{2k-1}(B) = h_0\mathfrak{S}_{2k-1}(B) \tag{22}$$

be its horizontal projection. This is given by the formula

$$
\begin{aligned}
&\mathcal{S}_{2k-1}(B) = k \int\limits_0^1 \mathcal{P}_{2k}(t,B)dt, \\
&\mathcal{P}_{2k}(t,B) = b_{r_1\ldots r_k}(a^{r_1}_{\mu_1} - B^{r_1}_{\mu_1})dx^{\mu_1} \wedge \mathcal{F}^{r_2}(t,B) \wedge \cdots \wedge \mathcal{F}^{r_k}(t,B), \\
&\mathcal{F}^{r_j}(t,B) = \frac{1}{2}[ta^{r_j}_{\lambda_j\mu_j} + (1-t)\partial_{\lambda_j}B^{r_j}_{\mu_j} - ta^{r_j}_{\mu_j\lambda_j} - (1-t)\partial_{\mu_j}B^{r_j}_{\lambda_j}) + \\
&\qquad \frac{1}{2}c^{r_j}_{pq}(ta^p_{\lambda_j} + (1-t)B^p_{\lambda_j})(ta^q_{\mu_j} + (1-t)B^q_{\mu_j}]dx^{\lambda_j} \wedge dx^{\mu_j} \otimes e_r.
\end{aligned}
$$

Now, let us consider the CS gauge model on a $(2k-1)$-dimensional base manifold X whose Lagrangian

$$L_{\mathrm{CS}} = \mathcal{S}_{2k-1}(B) \tag{23}$$

is the CS form (22) on J^1C. Clearly, this Lagrangian is not gauge-invariant. Let ξ_C (13) be the infinitesimal generator of gauge transformations of the connection bundle C. Its jet prolongation onto J^1C is

$$J^1\xi_C = \xi^r_\mu \partial^\mu_r + d_\lambda \xi^r_\mu \partial^{\lambda\mu}_r.$$

The Lie derivative of the Lagrangian L_{CS} along $J^1\xi_C$ reads

$$\mathbf{L}_{J^1\xi_C}\mathcal{S}_{2k-1}(B) = J^1\xi_C\rfloor d(h_0\mathfrak{S}_{2k-1}(B)) = \mathbf{L}_{J^1\xi_C}(h_0\mathfrak{S}_{2k-1}(B)). \tag{24}$$

A direct computation shows that

$$
\begin{aligned}
&\mathbf{L}_{J^1\xi_C}(h_0\mathfrak{S}_{2k-1}(B)) = h_0(\mathbf{L}_{\xi_C}\mathfrak{S}_{2k-1}(B)) = h_0(\xi_C\rfloor d\mathfrak{S}_{2k-1}(B) + d(\xi_C\rfloor \mathfrak{S}_{2k-1}(B)) = \\
&\qquad h_0(\xi_C\rfloor d\mathfrak{S}_{2k-1}(B) + d_H(h_0(\xi_C\rfloor \mathfrak{S}_{2k-1}(B)).
\end{aligned}
$$

By virtue of the transgression formula (19), we have

$$d(\xi_C\rfloor d\mathfrak{S}_{2k-1}(B)) = \mathbf{L}_{\xi_C}(d\mathfrak{S}_{2k-1}(B)) = \mathbf{L}_{\xi_C}P_{2k}(\mathfrak{F}) = 0.$$

It follows that $\xi_C\rfloor d\mathfrak{S}_{2k-1}(B)$ is a closed form on C, i.e.,

$$\xi_C\rfloor d\mathfrak{S}_{2k-1}(B) = d\psi + \varphi,$$

where φ is a non-exact $(2k-1)$-form on X. Moreover, $\varphi = 0$ since $P(\mathfrak{F})$, $k > 1$, does not contain terms linear in da^r_μ. Hence, the Lie derivative (24) takes the form (10) where

$$\mathbf{L}_{J^1\xi_C}\mathcal{S}_{2k-1}(B) = d_H\sigma, \qquad \sigma = h_0(\psi + \xi_C\rfloor \mathfrak{S}_{2k-1}(B)).$$

As a consequence, CS theory with the Lagrangian (23) admits the conservation law (11).

In a more general setting, one can consider the sum of the CS Lagrangian (23) and some gauge-invariant Lagrangian. For instance, let G be a semi-simple group and a^G the Killing form on $\mathfrak{g}$. Let

$$P(\mathfrak{F}) = \frac{h}{2} a^G_{mn} \mathfrak{F}^m \wedge \mathfrak{F}^n \tag{25}$$

be the second Chern form up to a constant multiple. Given a section B of $C \to X$, the transgression formula (19) on C reads

$$P(\mathfrak{F}) - P(F_B) = d\mathfrak{S}_3(B), \tag{26}$$

where $\mathfrak{S}_3(B)$ is the CS 3-form up to a constant multiple. Let us consider the gauge model on a 3-dimensional base manifold whose Lagrangian is the sum $L = L_{\rm CS} + L_{\rm inv}$ of the CS Lagrangian

$$\begin{aligned} L_{\rm CS} = h_0(\mathfrak{S}_3(B)) = [&\frac{1}{2} h a^G_{mn} \varepsilon^{\alpha\beta\gamma} a^m_\alpha (\mathcal{F}^n_{\beta\gamma} - \frac{1}{3} c^n_{pq} a^p_\beta a^q_\gamma) \\ &- \frac{1}{2} h a^G_{mn} \varepsilon^{\alpha\beta\gamma} B^m_\alpha (F(B)^n_{\beta\gamma} - \frac{1}{3} c^n_{pq} B^p_\beta B^q_\gamma) - d_\alpha(h a^G_{mn} \varepsilon^{\alpha\beta\gamma} a^m_\beta B^n_\gamma)] d^3x, \\ \mathcal{F} = h_0 \mathfrak{F} = \frac{1}{2} &\mathcal{F}^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r, \qquad \mathcal{F}^r_{\lambda\mu} = a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu, \end{aligned} \tag{27}$$

and some gauge-invariant Lagrangian

$$L_{\rm inv} = \mathcal{L}_{\rm inv}(x^\lambda, a^r_\mu, a^r_{\lambda\mu}, z^A, z^A_\lambda) d^3x \tag{28}$$

of gauge potentials a and matter fields z. Then, the first variational formula (6) on-shell takes the form

$$\mathbf{L}_{J^1\xi_C} L_{\rm CS} \approx d_H(\mathfrak{J}_{\rm CS} + \mathfrak{J}_{\rm inv}), \tag{29}$$

where $\mathfrak{J}_{\rm CS}$ is the Noether current of the CS Lagrangian (27) and $\mathfrak{J}_{\rm inv}$ is that of the gauge-invariant Lagrangian (28). A simple calculation gives

$$\begin{aligned} &\mathbf{L}_{J^1\xi_C} L_{\rm CS} = -d_\alpha(h a^G_{mn} \varepsilon^{\alpha\beta\gamma} (\partial_\beta \xi^m a^n_\gamma + (\partial_\beta \xi^m + c^m_{pq} a^p_\beta \xi^q) B^n_\gamma)) d^3x, \\ &\mathfrak{J}^\alpha_{\rm CS} = h a^G_{mn} \varepsilon^{\alpha\beta\gamma} (\partial_\beta \xi^m + c^m_{pq} a^p_\beta \xi^q)(a^n_\gamma - B^n_\gamma). \end{aligned}$$

Substituting these expressions into the weak equality (29), we come to the conservation law

$$0 \approx d_\alpha[h a^G_{mn} \varepsilon^{\alpha\beta\gamma} (2\partial_\beta \xi^m a^n_\gamma + c^m_{pq} a^p_b a^n_\gamma \xi^q) + \mathfrak{J}^\alpha_{\rm inv}]$$

of the modified Noether current

$$\overline{\mathfrak{J}} = ha^G_{mn}\varepsilon^{\alpha\beta\gamma}(2\partial_\beta\xi^m a^n_\gamma + c^m_{pq}a^p_b a^n_\gamma\xi^q) + \mathfrak{J}^\alpha_{\rm inv}.$$

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